A Novel Algorithm for Hierarchical State and Parameter Estimation in Slowly Time Varying Systems

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Abstract

Keywords: Legendre basis functions, Parameter estimation, State estimation, State space model, Slowly time varying systems.

This paper proposes a new hierarchical technique of state and parameter estimation for slowly time-varying systems. Slowly time-varying systems are considered to be the most tractable time-varying systems whose behavior is similar to time invariant systems over a small period of time. In this paper, a new method of parameter estimation is provided for slowly time-varying systems based on Legendre basis functions. Then, the states of the system are achieved through the Kalman filter using the estimated parameters. An important advantage of using Legendre basis functions is that identification of both slow and fast varying systems can be handled simultaneously. Furthermore, a considerable reduction in the number of parameters essential to follow each time-varying coefficient can be achieved using this method. In this work, the input signal to the time-varying system is chosen to be a pseudo random binary sequence (PRBS) which is an applicable signal in practice. Finally, an important feature of the proposed method is that states and parameters are estimated simultaneously. Effectiveness of the proposed method is shown through numerical simulations.

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1. Introduction

Many physical systems are naturally time-varying. Time-varying systems appear due to the impermanent operating conditions of system mechanism [1-5]. One property of time-varying processes is that such systems contain non-stationary transient behaviors [3]. One of the most obedient time-varying systems is the slowly time-varying ones whose behavior is similar to time invariant systems over a small period of time. Slowly time-varying systems are of vast importance in both practical applications and theoretical studies. Environmental situation variations are usually much slower than internal state or system dynamics. Thereupon, dynamic systems with parameters dependent on environment (e.g. temperature, pressure, etc) can often be modeled as slowly varying systems. Component aging is another example of slowly time-varying systems in operation [4]. As a result, when the parameters variations are satisfactorily smooth, these systems become slowly varying.

Parameter identification of time-varying systems is possible if each time-varying coefficient can be expressed using a linear or non-linear combination of a finite set of basis functions. The basis functions approach is based on an explicit model of parameter evolution. The most important advantage of using basis functions is that a considerable reduction in the number of parameters required to follow each time-varying coefficient can be achieved. In addition, identification of both slow
and fast varying systems can be handled by the use of basis functions [1], [6-11].

The most significant approach to identify a time-varying system is to apply an adaptive algorithm, assuming that the time-variations are slow. The popularity of adaptive algorithm is an indication of the prestige of time-varying system identification. Despite of their extensive utilize, adaptive algorithms cannot handle fast varying systems. If the coefficients change too fast compared with the algorithm’s convergence time, the adaptive algorithm will not be able to follow the system’s time evolution. Therefore, this is a blind spot for these algorithms. To overcome this problem, more explicit modeling of the coefficient’s variation is required. One approach for coefficients estimation is regarding them as stochastic processes. Then, the coefficients are estimated using the Kalman filter [12-21]. Least square algorithms have also been proposed to solve the parameter estimation problem [3], [22-25]. A least square numerical parameter estimation technique has been discussed in the field of state space system identification assuming that the states of the system are existing [26-35].

Recently, a parameter estimation technique was discussed for dynamic systems in which parameters are estimated based a recursive least squares technique first and then, the states of the system are computed through the Kalman filter using the estimated parameters.

This paper proposes an identification method for canonical state space systems in which the states are un-accessible and there are slowly time-varying parameters. Furthermore, it has been assumed that an explicit model of parameter evolution is known beforehand. A hierarchical approach is used in this paper for identification of slowly time-varying systems in which the parameters are identified using Legendre basis functions first and then, the states of the system are computed through the Kalman filter using the estimated parameters.

The remainder of this paper has the following structure: The identification model for state space systems is given first in Section 2. Afterwards, Section 3 introduces the basis functions approach and Section 4 gives the parameter and state estimation algorithm. Then, section 5 provides numerical simulations to confirm the efficiency of the proposed algorithm. Finally, conclusions are given in section 6.

2. The identification model for the time varying systems

Consider the following canonical state space system:

\[ x(t+1) = A(t)x(t) + b(t)u(t) \]
\[ y(t) = cx(t) + v(t), \]

where \( x(t) \in R^n \) is the state vector, \( u(t) \in R \) is the system input, \( y(t) \in R \) is the system output, \( v(t) \in R \) is measurement noise with zero mean, \( A(t) \in R^{n \times n} \), \( b(t) \in R^n \) and \( c \in R^{1 \times n} \) are the system parameter matrix and vectors, respectively, which are defined as follows:

\[
A(t) = \begin{bmatrix}
-a_1(t) & 1 & 0 & \ldots & 0 \\
-a_2(t) & 0 & 1 & \ldots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-a_n(t) & 0 & \ldots & 0 & 1 \\
\end{bmatrix},
\]

\[
b(t) = \begin{bmatrix}
b_1(t) \\
b_2(t) \\
\vdots \\
b_{n-1}(t) \\
b_n(t) \\
\end{bmatrix},
\]

\[
c = [1 \ 0 \ 0 \ \ldots \ 0].
\]

According to the above information and notations, and with expanding equation (1), we have:

\[
x_i(t+1) = -a_1(t)x_1(t) + \ldots + x_i(t) + a_{i-1}(t)x_{i-1}(t) + b_i(t)u(t), \quad i = 1, 2, \ldots, n,
\]

\[
y(t) = x_1(t) + v(t), \quad x_1(t) = 0, \quad i > n.
\]
Multiplying (3) by $z^{-i}$ and summing for $i=1,2,...,n$ results in [30]:

\[ x_i(t) = \sum_{i=1}^{n} a_i(t)x_i(t-i) + \sum_{i=1}^{n} b_i(t)u(t-i). \]  

(4)

By substituting (4) in (3), the regression equation or the time-varying autoregressive with an exogenous (TVARX) model of this system is achieved [29, 30], [35]:

\[ y(t) = \chi^T(t)\theta(t) + \nu(t), \]  

(5)

where the parameter vector $\theta(t) \in R^{2n}$ and the regression vector $\chi(t) \in R^{2n}$ are as follows:

\[ \theta(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad \chi(t) = \begin{bmatrix} -\eta -n(t-1) \\ -\eta -n(t-2) \\ \vdots \\ -\eta -n(t-n) \\ u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n) \end{bmatrix}. \]  

(6)

This paper deals with the identification of parameters $a_i(t)$ and $b_i(t)$ using basis functions approach.

3. Basis functions approach

A set of linearly independent functions, called the basis functions, are defined by:

\[ \{ B_j(t), j=1,...,m \}, \]  

(7)

where $m$ is the expansion dimension. If there is some prior information about the coefficient’s evolution, one may select the basis functions accordingly. Otherwise, any general linearly independent basis function can be used. Assume that each time-varying parameter $a_i(t)$ and $b_i(t)$ of parameter vector $\theta(t)$ in (5) can be expanded as a linear combination of these basis functions. As a result:

\[ \theta_i(t) = \sum_{j=1}^{m} \rho_{ij} B_j(t). \]  

(8)

where $i=2,...,2n$. By expanding parameter vector $\theta(t)$ as in (8), we can express $\chi(t)$ in (5) as the generalized regression vector $\Psi(t)$ associated with the analyzed time-varying process,

\[ \Psi(t) = \chi(t) \otimes B(t) = \begin{bmatrix} \chi_1(t)B_1(t) \\ \vdots \\ \chi_{2n}(t)B_{2n}(t) \end{bmatrix}^T. \]  

(9)

where $B(t) = \begin{bmatrix} B_1(t) \\ \vdots \\ B_{2n}(t) \end{bmatrix}$ is the basis functions vector and $\chi_i(t)$ is the $i$th component of $\chi(t)$, and $\chi(t) \otimes B(t)$ is the Kronecker product which is defined as follow:

**Definition 1.** For two matrices $A \in R^{k \times l}$ and $B \in R^{p \times q}$, the direct kronecker product, written as $A \otimes B$ is defined as [23]:

\[ A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1l}B \\ a_{21}B & a_{22}B & \cdots & a_{2l}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}B & a_{k2}B & \cdots & a_{kl}B \end{bmatrix} \in R^{kp \times lq}. \]  

(10)

Now let $\gamma = \begin{bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \\ \gamma_{2n}^T \end{bmatrix}$, be the vector of all coefficients used in equation (8). Using the generalized regression vector $\Psi(t)$, the system in (7) can be written as:

\[ y(t) = \Psi^T(t)\gamma + \nu(t). \]  

(11)

Since in the new regression equation (11), the measurement noise is white, the estimation of $\gamma$ can be achieved using the technique of recursive least square.

3.1. Legendre basis function

In this section, it is assumed that some prior information about the coefficient’s evolution is existing. Therefore, three Legendre basis functions can be given by [1]:

\[ B_1(t) = \frac{1}{\sqrt{S}}, \quad B_2(t) = \frac{S(S-1)}{S(S+1)} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ t-1 \end{bmatrix}, \quad \text{and} \quad B_3(t) = \frac{S(S-1)(S-2)}{S(S+1)(S+2)} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ t-1 \\ \frac{(t-1)(t-2)}{S-1} \end{bmatrix}. \]  

(12)
where $s$ is the number of samples. These basis functions are appropriate for identification of slowly time-varying systems, since they are continuously changeable.

4. The Parameter and State Estimation Techniques

In this section, estimating the states of the system through the Kalman filter is presented first. After that, estimating the parameters are discussed based on Legendre basis functions.

4.1. The state estimation

If the parameter matrix $\Lambda(t)$ and the parameter vector $b(t)$ are identified, then one may apply the following Kalman filter to calculate the estimate $\hat{x}(t)$ of the state vector $x(t)$:

$$\dot{x}(t+1) = \dot{x}(t) + \dot{b}(t) + k_x(t)[y(t) - c\hat{x}(t)]$$
$$k_x(t) = \Lambda(t)\mathbf{P}_s(t)\mathbf{c}^T[1 + \mathbf{c}\mathbf{P}_s(t)\mathbf{c}^T]^{-1}$$
$$\mathbf{P}_s(t+1) = \Lambda(t)\mathbf{P}_s(t)\Lambda^T(t) - k_x(t)c\mathbf{P}_s(t)\Lambda^T(t).$$

where $k_x(t)$ is the filter gain vector and $\mathbf{P}_s(t)$ is covariance matrix for the state estimation step. In the proposed method, the parameters are estimated based on Legendre basis functions first. After estimating the parameter matrix $\Lambda(t)$ and parameter vector $b(t)$, the states of the system are achieved through the Kalman filter using the estimated parameters. Therefore, the estimated parameter matrix $\hat{\Lambda}(t)$ and the parameter vector $\hat{b}(t)$ are used to compute the state vector estimate $\hat{x}(t)$ [20, 21].

$$\dot{x}(t+1) = \dot{\hat{x}}(t) + \dot{\hat{b}}(t) + k_x(t)[y(t) - c\hat{x}(t)]$$
$$k_x(t) = \hat{\Lambda}(t)\mathbf{P}_s(t)\mathbf{c}^T[1 + \mathbf{c}\mathbf{P}_s(t)\mathbf{c}^T]^{-1}$$
$$\mathbf{P}_s(t+1) = \hat{\Lambda}(t)\mathbf{P}_s(t)\hat{\Lambda}^T(t) - k_x(t)c\mathbf{P}_s(t)\hat{\Lambda}^T(t).$$

4.2. The parameter estimation

Let $\hat{\theta}(t)$ represent the estimate of $\theta$ at time $t$.

According to the above explanations, first the estimation of $\hat{\gamma}$ can be obtained from (11) using the method of recursive least square with forgetting factor.

$$\dot{\hat{\gamma}}(t) = \dot{\gamma}(t-1) + k_p(t)[y(t) - \hat{\eta}(t)\hat{\gamma}(t-1)]$$
$$k_p(t) = \frac{\mathbf{P}_p(t-1)\eta(t)}{\lambda + \eta^T(t)\mathbf{P}_p(t-1)\eta(t)}$$
$$\mathbf{P}_p(t) = \frac{1}{\lambda}\left[\mathbf{P}_p(t-1) - \frac{\mathbf{P}_p(t-1)\eta(t)\eta^T(t)\mathbf{P}_p(t-1)}{\lambda + \eta^T(t)\mathbf{P}_p(t-1)\eta(t)}\right].$$

where $k_p(t)$ is the filtering gain, $\mathbf{P}_p(t)$ is the estimation covariance matrix for the parameter estimation step, and $\lambda$ is the forgetting factor. Finally, according to (8), the estimated parameter vector $\hat{\theta}(t)$ is achieved using the estimated vector $\hat{\gamma}(t)$. In this paper, indices $s$ and $p$ are coordinate with the state and parameter respectively.

The algorithm of state and parameter estimation can be described in detail as:

- Initialization based on prior information and let $t=1$.
- Assemble the input-output data $u(t)$ and $y(t)$.
- Form the $\eta(t)$ using (6).
- Compute the gain vector $k_p(t)$ and the covariance matrix $\mathbf{P}_p(t)$ for parameter estimation step using the equation (15), and according to (8), the parameter vector estimate $\hat{\theta}(t)$ is achieved using the estimated vector $\hat{\gamma}(t)$.
- Find out $\hat{\eta}(t)$ and $\hat{\gamma}(t)$ from the estimated parameter $\hat{\theta}(t)$ according to (6), and formulate $\hat{\Lambda}(t)$ and $b(t)$ using (2).
- Compute the filter gain vector $k_x(t)$ and the covariance matrix $\mathbf{P}_s(t)$ for state estimation step using (14), and update the state estimate $\hat{x}(t+1)$.
- Set $t=t+1$.

The proposed identification method is not limited to using Legendre basis functions and other basis
functions can also be used including Fourier sequences, Walsh functions and wavelets [5-8], [36, 37]. The system identification loop of computing the parameter estimate $\hat{\theta}(t)$ and the state estimate $\hat{x}(t+1)$ is shown in Figure 1.

Fig. 1: The system identification loop.

5. Numerical Simulations

The input $u(t)$ is chosen to be a pseudo random binary sequence (PRBS), and $v(t)$ is a white noise sequence with zero mean. In these examples $v(t)$ is a white noise sequence with zero mean, and variances 0.01 and 1. Also, in these simulations, the initial values are chosen as $P_{p0} = P_{s0} = 10^3 \cos(12), \gamma_0 = P_{p0}^{-1} (..,1).

Example 1: Consider the following state space time varying system:

$$x(t + 1) = \begin{bmatrix} 0.5\sin(0.04\pi t) & 1 \\ -0.3\sin(0.05\pi t) & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1.68\sin(0.04\pi t) \\ 2.32\sin(0.05\pi t) \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v(t).$$

Assume that the time-varying parameter vector (to be estimated), is given as:

$$\theta(t) = [\alpha_1(t), \alpha_2(t), b_1(t), b_2(t)]^T = \begin{bmatrix} -0.5\sin(0.04\pi t) \\ 0.3\sin(0.05\pi t) \\ 1.68\sin(0.04\pi t) \\ 2.32\sin(0.05\pi t) \end{bmatrix}$$

The parameter estimates $\hat{\alpha}_1(t)$ and $\hat{\alpha}_2(t)$ are shown versus $t$ in Figures 2, 3, respectively. Figure 2 shows the parameters estimates using Legendre basis functions approach with $\lambda = 0.89$ and variance 1. Figure 3 shows the estimated parameters with $\lambda = 0.97$ and variance 0.01.

Example 2: Consider the following state space time varying system:

$$x(t + 1) = \begin{bmatrix} -2.5\sin(0.04\pi t) & 1 \\ 7\sin(0.05\pi t) & 0 \end{bmatrix} x(t) + \begin{bmatrix} -0.93\sin(0.04\pi t) \\ -4.5\sin(0.05\pi t) \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v(t).$$

Assume that the time-varying parameter vector (to be estimated), is given as:

$$\theta(t) = [\alpha_1(t), \alpha_2(t), b_1(t), b_2(t)]^T = \begin{bmatrix} 2.5\sin(0.04\pi t) \\ -7\sin(0.05\pi t) \\ -0.93\sin(0.04\pi t) \\ -4.5\sin(0.05\pi t) \end{bmatrix}$$

The parameter estimates $\hat{\alpha}_1(t)$ and $\hat{\alpha}_2(t)$ are shown versus $t$ in Figures 4, 5, respectively. Figure 4 shows the parameters estimates using Legendre basis functions approach with $\lambda = 0.89$ and variance 1. Figure 5 shows the estimated parameters with $\lambda = 0.97$ and variance 0.01. The simulations results indicate that the proposed algorithms are successful.
Fig. 2: The time evolution of parameters with $\lambda = 0.89$ and variance=1.
Fig. 3: The time evolution of parameters with $\lambda = 0.97$ and variance=0.01.
Fig. 4: The time evolution of parameters with $\lambda = 0.89$ and variance=1.
Fig. 5: The time evolution of parameters with $\lambda = 0.97$ and variance=0.01.
4. Conclusions

This paper proposed a novel algorithm for state and parameter estimation in slowly time varying systems. In this paper, parameters of a time-varying autoregressive model with exogenous input (TVARX) are estimated using Legendre basis functions approach. The simulation results indicate the promising performance of the proposed method. The proposed parameter identification method is not limited to using Legendre basis functions and other basis functions can be used instead. Finally, an important feature of the proposed method is that states and parameters are estimated simultaneously. Effectiveness of the proposed method is shown through numerical simulations.

References


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